Signatures in representations of rational Cherednik algebras

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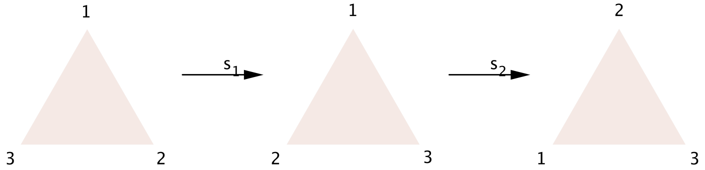
- $\partial_1(x_1 \cdot x_1^n) - x_1 \cdot \partial_1(x_1^n) = (n+1) \cdot x^n - n \cdot x^n = 1 \cdot x^n$
- $\partial_1 x_1 = x_1 \partial_1 + 1$
- $\partial_1 x_2 = x_2 \partial_1$
- $\partial_2 x_1 = x_1 \partial_2$
- $\partial_2 x_2 = x_2 \partial_2 + 1$

- D_n: symmetries of regular n-gon
- Rotations r_{theta}, reflections s_j
- D_3: $s_2 \circ s_1 = r_{120}$



- Unit normal vectors

$$\left(-\sin\left(\frac{2\pi j-2\pi}{n}\right),\cos\left(\frac{2\pi j-2\pi}{n}\right)\right)$$

Defining the rational Cherednik algebra

- $\mathfrak{h} = \text{span}(\partial_1, \partial_2)$
- $\mathfrak{h}^* = \text{span}(x_1, x_2)$
- S of reflections $s \in D_n$
- $\alpha_s \in \mathfrak{h}^*$ unit normal
- $\alpha_s^\vee \in \mathfrak{h}$

Defining the rational Cherednik algebra

For $x, x' \in \mathfrak{h}^*$, $\partial, \partial' \in \mathfrak{h}$, $s \in S$:

$$sxs^{-1} = s(x), \quad s\partial s^{-1} = s(\partial), \quad [x, x'] = 0, \quad [\partial, \partial'] = 0$$

$$[\partial, x] = (\partial, x) - \sum_{s \in S} \kappa(s)(\partial, \alpha_s)(\alpha_s^\vee, x)s$$

The rational Cherednik algebra over D_3
Example

$$\begin{aligned} \partial_1 x_1 &= x_1\partial_1 + [\partial_1, x_1] \\ &= x_1\partial_1 + (\partial_1, x_1) - \sum_{s\in S} \kappa(\partial_1, \alpha_s)(\alpha_s^\vee, x_1)s \\ &= x_1\partial_1 + 1 - \frac{3}{2}\kappa s_2 - \frac{3}{2}\kappa s_3 \end{aligned}$$

The trivial representation

- $\partial \to 0$
- $s \to 1$
- $x \to x$

Signatures

- $\langle x+y, z+w\rangle = \langle x, z\rangle + \langle x, w\rangle + \langle y, z\rangle + \langle y, w\rangle$
- $\langle ax, by\rangle = \bar{a}b\langle x, y\rangle$,
- $\langle w, z\rangle = \overline{\langle z, w\rangle}$.
- $\langle 1, 1\rangle = 1$
- $\langle x_1 x, x'\rangle = \langle x, \partial_1 x'\rangle$
- $\langle x_2 x, x'\rangle = \langle x, \partial_2 x'\rangle$.

Choosing a basis

- $\alpha = \frac{x_1 - ix_2}{\sqrt{2}}$
- $\bar{\alpha} = \frac{x_1 + ix_2}{\sqrt{2}}$
- $\beta = \frac{\partial_1 + i\partial_2}{\sqrt{2}}$,
- $\bar{\beta} = \frac{\partial_x - i\partial_2}{\sqrt{2}}$.

Choosing a basis

$$\begin{aligned}
\langle \alpha_1 x, x' \rangle &= \langle \tfrac{(x_1 - i x_2)}{\sqrt{2}} x, x' \rangle \\
&= \langle \tfrac{x_1}{\sqrt{2}} x, x' \rangle + \langle -\tfrac{i x_2}{\sqrt{2}} x, x' \rangle \\
&= \langle \tfrac{x}{\sqrt{2}}, \partial_1 x' \rangle + \langle x, \tfrac{i \partial_2}{\sqrt{2}} x' \rangle \\
&= \langle x, (\tfrac{\partial_1 + i \partial_2}{\sqrt{2}}) x' \rangle \\
&= \langle x, \beta_1 x' \rangle
\end{aligned}$$

Choosing a basis

$$s(\alpha) = e^{-2i\theta(s)}\bar{\alpha},$$
$$s(\bar{\alpha}) = e^{2i\theta(s)}\alpha,$$
$$s(\beta) = e^{-2i\theta(s)}\bar{\beta}$$
$$s(\bar{\beta}) = e^{2i\theta(s)}\beta$$
$$[\beta, \alpha] = 1 - \sum_{s\in S} \kappa s,$$
$$[\beta, \bar{\alpha}] = \sum_{s\in S} \kappa e^{2i\theta(s)} s,$$
$$[\bar{\beta}, \alpha] = \sum_{s\in S} \kappa e^{-2i\theta(s)} s,$$
$$[\bar{\beta}.\bar{\alpha}] = 1 - \sum_{s\in S} \kappa s,$$

Signatures with D_3

$$\begin{aligned}
\langle \alpha, \alpha \rangle &= \langle 1, \beta\alpha \rangle \\
&= \langle 1, 1 - \sum_{s \in S} \kappa \rangle \\
&= 1 - 3\kappa \\
\langle \alpha, \bar{\alpha} \rangle &= \langle 1, \beta\bar{\alpha} \rangle \\
&= \langle 1, -\sum_{s \in S} \kappa \rangle \\
&= 0
\end{aligned}$$

Signatures with D_{2n+1}

$$
\begin{aligned}
\langle \alpha, \alpha \rangle &= 1 - \sum_{s \in S} \kappa \\
&= 1 - (2n+1)\kappa, \\
\langle \alpha, \bar{\alpha} \rangle &= 0.
\end{aligned}
$$

Hermitian form on D_{2n+1}

$$
\begin{aligned}
\langle \alpha^r \bar{\alpha}^{N-r}, \alpha^s \bar{\alpha}^{N-s} \rangle &= s \langle \alpha^{r-1} \alpha^{\bar{N}-r}, \alpha^{s-1} \bar{\alpha}^{N-s} \rangle \\
&\quad - \langle \alpha^{r-1} \bar{\alpha}^{N-r}, \sum_{j=1}^{S} \alpha^{N-j} \bar{\alpha}^{j-1} \sum \kappa e^{2i\theta(N-s-j+1)} \rangle \\
&\quad + \langle \alpha^{r-1} \bar{\alpha}^{N-r}, \sum_{j=1}^{N-s} \alpha^{N-j} \bar{\alpha}^{j-1} \sum \kappa e^{2i\theta(N-s-j+1)} \rangle
\end{aligned}
$$

Orthogonal Pairings

Theorem

If $2n+1 \nmid r-t$, then in the Hermitian form on the polynomial representation of $H_\kappa(D_{2n+1})$,

$$\langle \alpha^r \bar{\alpha}^{N-r}, \alpha^t \bar{\alpha}^{N-t} \rangle = 0,$$

where $N, r, t \in \mathbb{Z}^{\geq 0}$, $N \geq r$, $N \geq t$, $\alpha = x_1 - ix_2$, and $\bar{\alpha} = x_1 + ix_2$.

Basis element pairings

Theorem

If $N < 2n+1$ and $N - r \leq r$, then in the Hermitian form on the polynomial representation of $H_\kappa(D_{2n+1})$,

$$\langle \alpha^r \bar{\alpha}^{N-r}, \alpha^r \bar{\alpha}^{N-r} \rangle = (N-r)! \prod_{j=1}^{r} \left(j - \sum_{s \in S} \kappa \right),$$

where $N, r \in \mathbb{Z}^{\geq 0}$, $N \geq r$, $\alpha = x_1 - ix_2$, and $\bar{\alpha} = x_1 + ix_2$.

Further questions

- General description?
- 2n case?
  κ takes two different values.
- Other representations?
- Other reflection groups?